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HYPERSURFACES THROUGH HIGHER-CODIMENSIONAL SUBMANIFOLDS OF $\mathbb{C}^n$ WITH PRESERVED LEVI-KERNEL

BY

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ABSTRACT

For a "generic" submanifold S of a complex manifold X, we show that there exists a hypersurface $M \supset S$ which has the same number of negative (or positive) Levi-eigenvalues as S at one prescribed conormal (cf. also [9]). When $\operatorname{rank} L_S$ is constant, then M may be found such that L_M and L_S have the same number of negative eigenvalues at any common conormal. Assuming the existence of a hypersurface M with the above property, we then discuss the link between complex submanifolds of S whose tangent plane belongs to the null-space of the Levi-form L_S of S (of all complex submanifolds when L_S is semi-definite), and complex submanifolds of $\dot{T}^*_S X$. As an application we give a simple result on propagation of microanalyticity for CR-hyperfunctions along complex, L_S-null, curves (cf. [3]).

Let X be a complex manifold of dimension n, S a real C^2-submanifold of X of codimension l, T^*X the cotangent bundle to X, $T^*_S X$ the conormal bundle to S in X. Let $L_S(p), p \in \dot{T}^*_S X (= T^*_S X \setminus \{0\})$, be the Levi form of S with respect to p (cf. [5]), and denote by $s_S^{+,-,0}(p)$ the numbers of respectively positive, negative, and null eigenvalues of $L_S(p)$.

THEOREM 1: *Assume that S is generic (i.e. $TS + \sqrt{-1}TS = TX$), and fix $p_o \in \dot{T}^*_S X$. Then there exists a hypersurface M such that $M \supset S$, $\dot{T}^*_M X \ni p_o$, and moreover*

$$s_M^-(p_o) = s_S^-(p_o). \tag{1}$$

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Proof: We choose local coordinates $(z,\zeta) \in T^*X$, and, for $p_o = (w_o;\zeta_o)$, suppose ζ_o real. We write $||\zeta|| = (\sum_i \zeta_i^2)^{\frac{1}{2}}$ (for the determination of the square root which is positive for real ζ), fix $t \in \mathbb{R}^+$, and define a contact transformation $\chi = \chi_t$ by:

$$\chi\colon (z;\zeta) \mapsto \left(z - t\frac{\zeta}{||\zeta||};\zeta\right)$$

(where we suppose e.g. $\sum_i \zeta_i^2 \notin \mathbb{R}^-$). We have, for a hypersurface $\tilde{S} \subset X$,

$$\chi(\dot{T}^*_S X) = T^*_{\tilde{S}}X. \tag{2}$$

We shall use the notations $T^{\mathbb{C}}S = TS \cap \sqrt{-1}TS, \lambda_S = TT^*_S X$ and similarly for $\tilde{S}$. Since

$$\operatorname{Ker} L_S \underset{\pi'}{\overset{\sim}{\leftarrow}} \lambda_S \cap \sqrt{-1}\lambda_S \underset{\chi'}{\overset{\sim}{\rightarrow}} \lambda_{\tilde{S}} \cap \sqrt{-1}\lambda_{\tilde{S}} \underset{\pi'}{\overset{\sim}{\rightarrow}} \operatorname{Ker} L_{\tilde{S}},$$

and since $\dim(T^{\mathbb{C}}\tilde{S}) = \dim(T^{\mathbb{C}}S) + (l-1)$, then $\operatorname{rank}(L_{\tilde{S}}) = \operatorname{rank}(L_S) + (l-1)$. It follows, for t small enough, that

$$s^-_{\tilde{S}}(q) = s^-_S(p) + (l-1) \qquad (q = \chi(p)). \tag{3}$$

We note that we have an identification $\dot{T}^*_S X \xrightarrow{\sim} X \setminus S$ given by

$$(w;\zeta) \mapsto w - |\zeta|\frac{\zeta}{||\zeta||},$$

provided that z is close to S and $|\zeta|$ is small. We denote by h the projection $X \to S, z \mapsto w$. We also remark that we have

$$\lambda_S \cap \sqrt{-1}\lambda_S = \{(u;v)\colon v = v(u) = \partial\partial r u - \partial\bar{\partial} r u \text{ with } u \in \operatorname{Ker} L_S\},$$

and similarly for $\lambda_{\tilde{S}} \cap \sqrt{-1}\lambda_{\tilde{S}}$. Write $p_o = (w_o;\zeta_o), q_o (= \chi(p_o)) = (z_o;\zeta_o)$ with $\zeta_o \in \mathbb{R}^n$, $|\zeta_o| = 1$. Let us define a linear transformation on $\mathbb{C}^n$ by

$$\Phi_t\colon u \mapsto u - t(v(u) - \zeta_o\langle\zeta_o, v(u)\rangle).$$

The correspondence $\chi'_t(p_o)\colon \lambda_S(p_o) \cap \sqrt{-1}\lambda_S(p_o) \to \lambda_{\tilde{S}}(q_o) \cap \sqrt{-1}\lambda_{\tilde{S}}(q_o)$ induces a correspondence Φ_t: $\operatorname{Ker} L_S(w_o) \xrightarrow{\sim} \operatorname{Ker} L_{\tilde{S}}(z_o)$. We denote by $g\colon T_{z_o}X \to \tilde{S}$ the projection along the normal to $\tilde{S}$ at z_o, and write $R = g(\Phi_t(T_{z_o}S))$. We have

$$T^{\mathbb{C}}_{z_o}R \supset \operatorname{Ker} L_{\tilde{S}}(z_o), \quad T_{z_o}R = \Phi_t(T_{w_o}S).$$

Thus we may find a decomposition $T^{\mathbb{C}}_{z_o}\tilde{S} = T^{\mathbb{C}}_{z_o}R \oplus \tilde{N}$ such that

$$L_{\tilde{S}}(z_o)(\bar{u}, v) = 0 \quad \forall u \in T^{\mathbb{C}}_{z_o}R, \quad \forall v \in \tilde{N}. \tag{4}$$

We take now a hypersurface $\tilde{M}$ which intersects $\tilde{S}$ along R with order of contact 2 and with the property that, if $\tilde{M}^+$ and $\tilde{S}^+$ are the closed half-spaces with boundary $\tilde{M}, \tilde{S}$ and inward conormal q_o, then $\tilde{M}^+ \subset \tilde{S}^+$. This implies

$$\chi^{-1}(T^*_{\tilde{M}}X) = T^*_M X \quad \text{for a hypersurface } M \supset S. \tag{5}$$

We have

$$L_{\tilde{M}}(z_o)(\bar{u}, v) = 0 \quad \forall u \in T^{\mathbb{C}}R, \quad \forall v \in \tilde{N}. \tag{6}$$

This follows immediately from (4) if we notice that, since $\tilde{M} \cap \tilde{S} = R$, and $T^*_{\tilde{M}}X|_R = T^*_{\tilde{S}}X|_R$, then

$$L_{\tilde{M}}(z)(\bar{u}, \cdot)|_{T^{\mathbb{C}}\tilde{M}} = L_{\tilde{S}}(z)(\bar{u}, \cdot)|_{T^{\mathbb{C}}\tilde{S}} \quad \forall z \in R.$$

We also notice that $L_{\tilde{M}}(z)|_{T^{\mathbb{C}}R} = L_{\tilde{S}}(z)|_{T^{\mathbb{C}}R} \sim L_S(z)$ (for small t), and that, if $T^*_S X_{z_o}$ is identified with a totally real plane $N' \subset T_{z_o}X$ (by the Euclidean structure of $T_{z_o}X^{\mathbb{R}}$), then

$$L_{\tilde{M}}(z_o)(\bar{v}, v) \leq -ct^{-1}|v|^2 \quad \forall v \in N' \oplus \sqrt{-1}N'.$$

Note also that in the decomposition $T^{\mathbb{C}}_{z_o}\tilde{S} = T^{\mathbb{C}}_{z_o}R \oplus \tilde{N}$ which gives rise to (6), $\tilde{N}$ can be taken close to $N' \oplus \sqrt{-1}N'$ (for t small). It follows that

$$s^-_{\tilde{M}}(z_o) = s^-_S(p_o) + (l-1), \quad s^+_{\tilde{M}}(z_o) = s^+_S(p_o). \tag{7}$$

(Here $\tilde{M}$ being a hypersurface, we write $s^{-,+}_{\tilde{M}}(z_o)$ instead of $s^{-,+}_{\tilde{M}}(p_o)$.) On the other hand, from

$$\operatorname{Ker} L_{\tilde{M}} \underset{\pi'}{\overset{\sim}{\leftarrow}} \lambda_{\tilde{M}} \cap \sqrt{-1}\lambda_{\tilde{M}} \underset{\chi'^{-1}}{\overset{\sim}{\rightarrow}} \lambda_M \cap \sqrt{-1}\lambda_M \underset{\pi'}{\overset{\sim}{\rightarrow}} \operatorname{Ker} L_M,$$

we get

$$s^-_M(p_o) = s^-_{\tilde{M}}(z_o) - (l-1) = s^-_S(p_o),$$
$$s^+_M(p_o) = s^+_{\tilde{M}}(z_o) + (l-1) = s^+_S(p_o) + (l-1). \qquad \blacksquare$$

Note that (1) entails

$$\operatorname{Ker} L_S(p_o) \subset \operatorname{Ker} L_M(p_o). \tag{8}$$

Remark 2: Let $s_S^-(p) \equiv b$ for any p in a neighborhood of p_o on $\dot{T}_S^* X$. We tried to prove the existence of a hypersurface M containing S and verifying $s_M^-(p) \equiv b$ $\forall p \in \dot{T}_M^* X$ (in particular the existence of M pseudoconvex in case $s_S^-(p) \equiv 0 \; \forall p$). But we do not know the answer to this question yet.

THEOREM 3: *Let S be generic of class C^3 and* $\operatorname{rank} L_S(p) \equiv \text{const}$ $\forall p \in \dot{T}_S^* X$ *at p_o. Then there exists $M \supset S$ such that*

$$s_M^-(p) = s_S^-(p_o) \quad \forall p \in S \times_M \dot{T}_M^* X. \tag{9}$$

Proof: We transform $\chi(T_S^* X) = T_{\tilde{S}}^* X$ with $\chi = \chi_t, \tilde{S} = \tilde{S}_t$ as in Theorem 1. Our hypothesis is then equivalent to assuming that $L_{\tilde{S}}(z)$ has constant rank $\forall z$. Thus each $\tilde{S} = \tilde{S}_t$ is foliated by the complex integral leaves of $\operatorname{Ker} L_{\tilde{S}}$. (For this we require S of class C^3.) It follows that $X \setminus S$ itself is foliated by the integral leaves of $\operatorname{Ker} L_{\tilde{S}}$ for all values of the parameter t. We replace $R = g(\Phi_t(T_{w_o} S))$ by $f^{-1} f(R)$ where $f\colon \tilde{S} \to L$ ($L \subset \tilde{S}$ transversal to $\operatorname{Ker} L_{\tilde{S}}(z_o)$) is the projection along the integral leaves of $\operatorname{Ker} L_{\tilde{S}}$. We still have

$$T_z^{\mathbb{C}} R \supset \operatorname{Ker} L_{\tilde{S}}(z) \quad \forall z \in \mathbb{R}.$$

Then the line of the proof is the same as in Theorem 1. ∎

Remark 4: Assume that there exists a hypersurface M which contains S and such that

$$\operatorname{Ker} L_S(p) \subset \operatorname{Ker} L_M(p) \quad \forall p \in S \times_M \dot{T}_M^* X, \quad p \text{ close to } p_o. \tag{10}$$

(As we have already noticed in (8), this happens e.g. when $s_M^-(p) \equiv s_S^-(p) \forall p$.) Then for any equation $r = 0$ for M and for any complex $\gamma \subset S$ such that $T_z \gamma \subset \operatorname{Ker} L_S(p) \forall p = \partial r(z), \forall z \in \gamma$, we have

$$\left.\frac{\partial r}{\partial_{z_1} r}\right|_\gamma \text{ is complex.}$$

To prove this, we let $p_o = (1, 0, \ldots, 0)$, take an orthonormal system (e_i) in $\mathbb{C}^n$, assume $\gamma = \bigoplus_{i=n-d+1}^n \mathbb{C} e_i$, and put $w_i^j = \partial_{z_i} r e_j - \partial_{z_j} r e_i$. Then for any fixed j, $(w_i^j)_{i=1,\ldots,n, i \neq j}$ is a basis for $T_z^{\mathbb{C}} M$. It follows, $\forall h \geq n - d + 1$, that

$$\begin{aligned} \partial_{\bar{z}_h} \left(\frac{\partial_{z_i} r}{\partial_{z_1} r} \right) &= \left((\partial_{\bar{z}_h} \partial_{z_i} r)(\partial_{z_1} r) - (\partial_{\bar{z}_h} \partial_{z_1} r)(\partial_{z_i} r) \right) (\partial_{z_1} r)^{-2} \\ &= \left(\partial \bar{\partial} r(w_1^i, \bar{e}_h) \right) (\partial_{z_1} r)^{-2} = 0. \end{aligned} \tag{11}$$

The above remark has a valuable improvement:

THEOREM 5: *Let S be generic, assume (10), and also suppose*

$$s_S^-(p) \equiv \text{const} \quad \forall p \in \dot{T}_S^* X \text{ at } p_o. \tag{12}$$

Let γ be a complex submanifold of S such that

$$T_z\gamma \subset \operatorname{Ker} L_S(p) \quad \forall z \in \gamma, \quad \forall p \in (\dot{T}_S^* X)_z \text{ at } p_o. \tag{13}$$

Then there exists a unique r, with $r|_S \equiv 0, \partial r(z_o) = p_o$ such that $\partial r|_\gamma$ is complex.

Proof: Existence: According to [6] it is possible to interchange T_S^*X with $T_N^*X, \operatorname{codim} N = 1, s_N^-(q_o) = 0(q_o = \chi(p_o))$ by a contact transformation χ. Such a χ can be defined by

$$\chi : \begin{cases} z_1 \mapsto z_1 + \sqrt{-1}\sum_{j=1}^{n-1} \lambda_{1j} \dfrac{\zeta_j}{\zeta_n}, \\ \cdots\cdots, \\ z_n \mapsto z_n - \frac{\sqrt{-1}}{2} \sum_{ij=1}^{n-1} \lambda_{ij} \dfrac{\zeta_i\zeta_j}{\zeta_n^2}, \end{cases} \tag{14}$$
$$\zeta \mapsto \zeta,$$

for suitable (λ_{ij}). But in fact we have $s_N^-(z) \equiv 0 \ \forall z \in N$ because the constancy of s^- is invariant under contact transformation (cf. [4, ch. 11]); thus N is the boundary of a pseudoconvex domain (with outward conormal q_o). Let M be a hypersurface which contains S and satisfies (10), let $r = 0$ be an equation for M with $\partial r(z_o) = p_o$, and define $\tilde{\gamma} = \pi\chi(\partial r|_\gamma)$; we claim that $\tilde{\gamma}$ is complex. In fact (11) holds as a consequence of (10). Moreover we have

$$\begin{aligned} \partial_{\bar{z}_h} \left(\frac{\partial_{z_i} r \partial_{z_j} r}{(\partial_{z_1} r)^2} \right) \Big|_\gamma &= \big((\partial_{\bar{z}_h}\partial_{z_i} r)(\partial_{z_j} r)(\partial_{z_1} r) + (\partial_{\bar{z}_h}\partial_{z_j} r)(\partial_{z_i} r)(\partial_{z_1} r) \\ &\quad -2(\partial_{\bar{z}_h}\partial_{z_1} r)(\partial_{z_i} r)(\partial_{z_j} r)\big) (\partial_{z_1} r)^{-3} \\ &= \Big((\partial\bar{\partial} r(w_1^i, \bar{e}_h))(\partial_{z_j} r) + (\partial\bar{\partial} r(w_1^j, \bar{e}_h))(\partial_{z_i} r) \Big) (\partial_{z_1} r)^{-3} \\ &= 0, \end{aligned}$$

due to (10) and (13). At this point we apply [1] (with some minor modifications because now $\dim(\gamma)$ is possibly > 1) and find a complex section $t\partial r|_{\tilde{\gamma}} \subset T_N^* X$.

UNICITY: Let ∂r verify $\bar{\partial}\partial r|_\gamma \equiv 0, \partial r(z_o) = 0$; we aim to prove that $\partial r|_\gamma \equiv 0$. It is not restrictive to assume that γ is a disc in the complex $\mathbb{C}_{z_n}$-plane. Let $r_1 = 0, r_2 = 0, \ldots, r_l = 0$ ($l = \operatorname{codim}(S)$), be a system of independent equations for S, let $(a_i)_{1\leq i\leq l}$ be a real vector-valued function such that $\partial r = \sum_i a_i \partial r_i$, and choose coordinates such that the matrix $A\colon \stackrel{\text{def}}{=} (\partial_{z_i} r_j)_{1\leq i,j\leq l}$ is non-degenerate. Then (a_i) satisfies

$$\text{(15)} \qquad \begin{cases} \partial_{\bar{z}_n}(a_i)|_\gamma = (a_i)\left((\partial_{\bar{z}_n}A)A^{-1}\right)\Big|_\gamma, \\ \partial_{z_n}(a_i)|_\gamma = (a_i)\overline{((\partial_{\bar{z}_n}A)A^{-1})}\Big|_\gamma, \\ (a_i)(z_o) = 0. \end{cases}$$

Then by the unicity of the solution to (15) we get $(a_i) \equiv 0$. ∎

Any section γ^* of T_S^*X with $\pi\gamma^* = \gamma$ (such as $\partial r|_\gamma$) is called a **lift** of γ to T_S^*X. Thus if (10), (12) hold, then any complex manifold $\gamma \subset S$ which satisfies (13) has a unique complex lift γ^* in $S \times_M T_M^*X$. Note that, since

$$\text{(16)} \qquad T_p^{\mathbb{C}} T_S^*X \overset{\pi'}{\underset{\sim}{\to}} \operatorname{Ker} L_S(p),$$

then (13) is necessary for existence of a lift through any p close to p_o.

COROLLARY 6: *Let S be generic and assume*

$$\text{(17)} \qquad \operatorname{rank} L_S(p) \equiv \text{const} \quad \forall p \in \dot{T}_S^*X \text{ in a neighborhood of } p_o.$$

Let γ be a complex submanifold of S which satisfies (13). Then there exists unique complex submanifold $\gamma^* \subset \dot{T}_S^*X$ with $\pi\gamma^* = \gamma, \gamma^* \ni p_o$.

Proof: According to Theorem 3, (10) is satisfied for a suitable M. In our hypothesis ($s^\pm \equiv$ const), (12) is also satisfied. Then Theorem 5 applies. ∎

Note that when (17) holds, then the existence of a lift can be proved in an easier way than by Theorem 5. In fact let us interchange $\dot{T}_S^*X \overset{\chi}{\underset{\sim}{\to}} T_N^*X$, $\operatorname{codim} N = 1$, $s_N^- \equiv 0$ by a symplectic complex homogeneous transformation χ of type (14). Thus N is a peudoconvex hypersurface with $\operatorname{rank} L_N \equiv$ const. But then N is foliated by the integral leaves $\{\tilde{\Gamma}\}$ of $\operatorname{Ker} L_N$. We then apply [1] and find a foliation $\{\tilde{\Gamma}^*\}$ of T_N^*X with $\pi(\tilde{\Gamma}^*) = \Gamma$. This induces, via χ^{-1} and $\pi \circ \chi^{-1}$, a foliation $\{\Gamma^*\}$ and $\{\Gamma\}$ of T_S^*X and S, respectively. We note now that if Γ_p^* is the leaf through p, and $\Gamma_z (z = \pi(p))$ its projection, then (13) implies $\gamma \subset \Gamma_z$. Thus if we set

$$\gamma^*\colon \stackrel{\text{def}}{=} \Gamma_p^*|\gamma,$$

we get a complex lift of γ to $\dot{T}_S^*X$.

Remark 7: If in Theorem 5 we assume $s_{\bar{S}}^{-} \equiv 0$ instead of (12), we do not need to make the assumption $T_z\gamma \subset \operatorname{Ker} L_S(p)(z = \pi(p))\forall p$. In fact if u belongs to $T_z\gamma$, then $L_S(p)(u, \bar{u}) = 0, p \in \dot{T}_S^* X, u \in T_z\gamma$. Therefore when $L_S(p)$ is semidefinite, we get $L_S(p)(\cdot, \bar{u})|_{T^{\mathbb{C}}S} = 0$.

Remark 8: For the sake of completeness we give the outline of the proof of the quoted result by [1] with the suitable modifications. We assume $\tilde{\gamma} = \{0\} \times \cdots \times \mathbb{C}^d_{z''} \subset N$ with N pseudoconvex, take an equation $s = 0$ for N with $\partial s = q_o$ $(= \chi(p_o))$, and write $u = e_h (\in T\tilde{\gamma}), w_i^j = \partial_{z_i} s e_j - \partial_{z_j} s e_i$. We then have

$$L_s(z)(w, \bar{u}) = 0 \quad \forall w \in T_z^{\mathbb{C}} N, \quad \forall z \in \tilde{\gamma},$$

which implies

$$\partial_{\bar{z}_h} \left(\frac{\partial_{z_i} s}{\partial_{z_1} s} \right)(z) = \frac{L_s(z)(w_i^1, \bar{u})}{(\partial_{z_1} s(z))^2} = 0 \quad \forall z \in \tilde{\gamma}. \tag{18}$$

We claim that we can then find a real function $\mu = \mu(z'')$ such that

$$\partial_{\bar{z}_h} ((\partial_{z_1}(e^{\mu} s)|_{\tilde{\gamma}}) = 0 \quad \forall h. \tag{19}$$

If this is true, then by setting $\tilde{\gamma}^* = \{(z; e^{\mu(z)} \partial s(z)); z \in \tilde{\gamma}\}$, we get the conclusion. Let $q_o = (0; dy_1)$, and write

$$s = y_1 - x_1 a(z'', \bar{z}'') + O(|(z_2, \ldots, z_{n-d})|))(|z''|) + O(|(x_1, z_2, \ldots, z_{n-d})|^2).$$

It is immediate to check that the system (19) for real μ is equivalent to the system

$$\left(\partial_{\bar{z}_h} \mu + (\partial_{\bar{z}_h} a)/(a + \sqrt{-1}) = 0, \quad \partial_{z_h} \mu + (\partial_{z_h} a)/(a - \sqrt{-1}) = 0\right), \tag{20}$$

for complex μ. Now the compatibility conditions of (20) are

$$\partial^2_{z_h \bar{z}_k} a(1 + a^2) - 2a \partial_{z_h} a \partial_{\bar{z}_k} a = 0 \quad \forall h, k = n - d + 1, \ldots, n.$$

On the other hand, we have

$$L_s(w_h^1, \bar{w}_k^1) = \partial^2_{z_1 \bar{z}_1} s \partial_{z_h} s \partial_{\bar{z}_k} s + \partial^2_{z_h \bar{z}_k} s |\partial_{z_1} s|^2 - \left[\partial^2_{z_1 \bar{z}_k} s \partial_{z_h} s \overline{\partial_{z_1} s} + \partial^2_{\bar{z}_1 z_h} s \overline{\partial_{z_k} s} \partial_{z_1} s\right].$$

If we compute $L_s(w_h^1, \bar{w}_k^1)$ in $\tilde{S}$ for $z_2 = \cdots = z_{n-d} = 0$, we get

$$L_s(w_h^1, \bar{w}_k^1) = x_1 \left(\partial^2_{z_h \bar{z}_k} a(1 + a^2) - 2a \partial_{z_h} a \partial_{\bar{z}_k} a\right) + O(|x_1|^2).$$

Thus the coefficient of x_1 must be 0 because otherwise, for some $w \in \operatorname{Vect}\{w_h^1\}_{h = n-d+1, \ldots, n}$, $L_s(w, \bar{w})$ would change sign on N.

Example 9: If we choose $M \supset S$ such that $L_M(p)|_{T^{\mathbb{C}}S} \geq 0$ (i.e. $s_S^-(p) = 0$) $\forall p \in S \times_M \dot{T}_M^*X$ but $L_M(p) \not\geq 0$, then for a complex curve γ in S we might have no wish to find a complex *lift* γ^* in $\dot{T}_M^*X$. For example, let us consider in $\mathbb{C}^3$

$$S = \{x_1 = 0, y_3 = 2y_1y_2\}, \quad M = \{y_3 = z_1\bar{z}_2 + \bar{z}_1z_2\}, \quad p_o = (0; \mathrm{d}y_3).$$

Then $T_z^{\mathbb{C}}S = \mathbb{C}u$ where $u = (0, 1, 2\,\mathrm{Im}\, z_1)$ and therefore $L_S(p) \equiv 0 \forall p \in \dot{T}_S^*X$. On the other hand, S contains the complex curve $\gamma = \{0\} \times \mathbb{C}_{z_2} \times \{0\}$ but $\dot{T}_M^*X$ cannot contain any complex γ^*. Otherwise this latter would satisfy

$$T\gamma^* \subset T\dot{T}_M^*X \cap \sqrt{-1}T\dot{T}_M^*X (\underset{\pi'}{\simeq} \mathrm{Ker}\, L_M) = 0,$$

which is a contradiction.

Corollary 6 says, however, that we can choose another M with any prescribed conormal at $z_o = 0$, so that a complex *lift* to $\dot{T}_M^*X$ always exists. For example, with the preceding S and p_o a good choice for M is

$$M = \{z; y_3 + (z_1z_2 + \overline{z_1z_2}) = 0\}.$$

Let $\mathcal{C}_{S|X}$ and $\mathcal{B}_{S|X}$ be the complexes of respectively CR microfunctions and CR hyperfunctions along S. We recall that $\mathcal{B}_{S|X}$ is defined as $R\Gamma_S(\mathcal{O}_X)[l]$ (where $l = \mathrm{codim}_X\, M$ and $\mathcal{O}_X$ is the sheaf of holomorphic functions). When S is real analytic, $\mathcal{B}_{S|X}$ turns out to coincide with the tangential $\bar{\partial}$-complex over usual hyperfunctions $\mathcal{B}_{S|S^{\mathbb{C}}}$ ($S^{\mathbb{C}}$ = a complexification of S). Let sp: $H^0(\pi^{-1}\mathcal{B}_{S|X}) \to H^0(\mathcal{C}_{S|X})$ be the spectral morphism, and define

$$\mathrm{WF}(f) = \mathrm{supp}(\mathrm{sp}(f)), \quad f \in H^0(\mathcal{B}_{S|X}).$$

WF coincides with the usual analytic wave front set in the sense e.g. of [5].

The conormal along S to the hypersurface M which satisfies (10) describes the connection of $\dot{T}_S^*X$ in which the propagation of microanalyticity of CR-hyperfunctions takes place.

PROPOSITION 10: *Let S be generic and satisfy (10),(12), let γ be a complex curve of S, p_o a point of $\dot{T}_S^*X$ with $\pi(p_o) = z_o \in \gamma$, and suppose that $T_z\gamma \subset$ $\mathrm{Ker}\, L_S(p) \forall p \in (\dot{T}_S^*X)_z$. Then there exists a section $\partial r|_\gamma$ of $\dot{T}_S^*X$ with $\partial r(z_o) = p_o$, such that*

$$p_o \notin \mathrm{WF}(f)_{z_o} \text{ implies } \partial r(z) \notin \mathrm{WF}(f)_z \quad \forall f \in H^0(\mathcal{B}_{S|X}).$$

Proof: We choose an equation $r = 0$ for M, and consider the symplectic transformation χ of Theorem 5. Then $\pi\chi(\partial r|_\gamma)$ is a complex curve in the hypersurface N, the boundary of a pseudoconvex domain. According to [4], the sections of $\mathcal{C}_{S|X}$ are interchanged, by a quantization Φ_{K} of χ, with $\mathcal{H}^1_{N^+}(\mathcal{O}_X)$ where N^+ is the closed half-space with boundary N and inward conormal $q_o = \chi(p_o)$. Thus $p_o \notin \mathrm{WF}(f)$ if and only if $\Phi_{\mathrm{K}}(\mathrm{sp}(f))$ extends holomorphically across N at $\pi(q_o)$. On the other hand, one can check that the extendibility of a holomorphic function g across a hypersurface propagates along complex curves. To see this it is enough to use the submean property of the family of plurisubharmonic functions $\log|\partial_z^\alpha g|$, $\alpha \in \mathbb{N}^n$. ∎

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